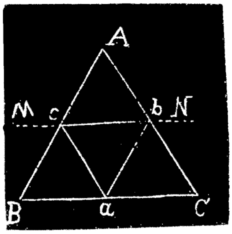
(*b*) If the angle-sum is less than a straight angle, the sum increases as the triangle grows less.

(*c*) If the angle-sum is greater than a straight angle, the sum decreases as the triangle grows less.

Solution by the PROPOSER.

Join the mid-points a, b, c, of the sides of an equilateral triangle ABC. The triangles with vertices A, B, C, being equal, we have $ab=bc=ac$, and $\angle Bac=\angle Cab=\angle Abc$, etc.



I. If $\angle A=\angle bac$, then $\angle A+\angle Abc+\angle Acb=\angle bac+Bac+\angle Cab=180°$, and it is easily shown that $\triangle abc=\triangle Abc$, etc.

Lemma to II and III.

If MN be a line moving towards coincidence with BC, so as always to cut off equal parts on AB, AC; then, according as the angle-sum of ABC is $<$ or $>$ 180°, so will that of $\triangle Abc$ be $<$ or $>$ 180°.

For otherwise, as the angle-sum of Abc would vary from $>$ or $<$ 180° to $<$ or $>$ 180°, there would be some position of MN in which the angle-sum of Abc would be 180°, with consequences incompatible with the hypothesis.

II. If $\angle A<60°$, then $\angle A+\angle Abc+\angle Acb<180°<\angle abc+\angle Abc+\angle Cba$.
$\therefore\ \angle abc>\angle A$.

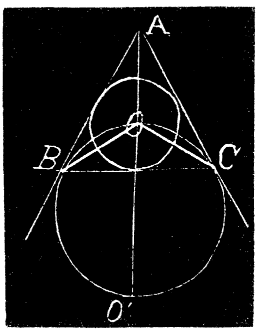
III. If $\angle A>60°$, then $\angle A+\angle Abc+\angle Acb>180°>\angle abc+\angle Abc+\angle Cba$.
$\therefore\ \angle abc<\angle A$.

162. Proposed by J. D. PALMER, Providence, Ky.

Given the distances from the vertices of a triangle, ABC, to the center of the in-circle, to construct the triangle.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

AO, BO, $CO=a$, b, c, respectively, where O is the center of the in-circle; BC, AC, $AB=x$, y, z, respectively. Let O_1 be the center of the ex-circle opposite A. Then



$$AO^2=\frac{(p-x)^2}{\cos^2\frac{1}{2}A},\ \text{where } p=\tfrac{1}{2}(x+y+z).$$

$$\therefore AO^2=\left(\frac{p-x}{p}\right)yz=yz-\frac{xyz}{p}=yz-4Rr=a^2.$$

$$\text{Similarly, } BO^2=\left(\frac{p-y}{p}\right)xz=xz-4Rr=b^2.$$

$$CO^2=\left(\frac{p-z}{p}\right)xy=xy-4Rr=c^2.$$

$$\therefore AO.BO.CO=abc=\frac{\triangle\, xyz}{p^2}=4Rr^2.$$

$\therefore$ $4Rr=abc/r$, and $(a^2r+abc)/r=yz$, $(b^2r+abc)/r=xz$, $(c^2r+abc)/r=xy$.

$$\therefore x(a^2r+abc)/r=y(b^2r+abc)/r=z(c^2r+abc)/r$$

$$=\sqrt{[(a^2r+abc)(b^2r+abc)(c^2r+abc)/r^3]}.$$

Also $a^2x+b^2y+c^2z=xyz\left(\frac{p-x}{p}+\frac{p-y}{p}+\frac{p-z}{p}\right)=xyz=1.$

$$\therefore \frac{a^2}{yz}+\frac{b^2}{xz}+\frac{c^2}{xy}=1, \text{ or } \frac{ar}{ar+bc}+\frac{br}{br+ac}+\frac{cr}{cr+ab}.$$

$\therefore$ $2abcr^2+(a^2b^2+a^2c^2+b^2c^2)r^2=a^2b^2c^2$.

$\therefore$ $r^3+Ar^2=B$. Let $r=S-\frac{1}{3}A$.

$\therefore$ $S^3-\frac{1}{3}A^2S=B-2A^3/27$.

$$\therefore S=\left(\tfrac{1}{2}B-\tfrac{1}{27}A^3+\sqrt{\frac{B^2}{4}-\frac{A^3B}{27}}\right)^{\frac{1}{3}}+\left(\tfrac{1}{2}B-\tfrac{1}{27}A^3-\sqrt{\frac{B^2}{4}-\frac{A^3B}{27}}\right)^{\frac{1}{3}}.$$

This determines r and therefore x, y, z, the sides of the triangle.

Otherwise, draw AO and produce AO to O_1 so that $OO_1=bc/r$. Upon OO_1 as diameter describe a circle. With O as center and b as a radius describe an arc cutting the circle in B. Similarly, with O as center and c as radius, draw an arc cutting the circle in C. Join BC, AC, AB, then ABC is the triangle required. For O_1 is the ex-center opposite A by construction as follows:

$$AO.AO_1=yz=(AO^2r+AO.BO.CO)/r.$$

$$\therefore AO_1=(AOr+BO.CO)/r.$$

The solution published in the last issue contained an error in the fourth line, and this vitiated the whole solution. ED.

163. Proposed by J. C. NAGLE, Professor of Civil Engineering in the Agricultural and Mechanical College of Texas, College Station, Tex.

Given the equal sides of an isosceles triangle and the radius of the inscribed circle to solve the triangle. As a numerical example let the known sides be 27 and the radius of the inscribed circle 3.5. The problem occured in connection with some mill work and the exterior angles of the triangle were required in order to make patterns for iron braces.